On 3-dimensional asymptotically harmonic manifolds with minimal horospheres

Hemangi Shah*

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Department of Mathematics, Indian Institute of Science,
Bangalore - 560012, India.

Abstract

Let (M,g) be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. We show that M is a flat manifold, provided M is asymptotically harmonic of constant $h = 0$.

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1 Introduction

Let (M,g) be a complete, simply connected Riemannian manifold without conjugate points. Let SM be the unit tangent bundle of M. For $v \in SM$, let γ_v be the geodesic with $\gamma_v'(0) = v$ and $b_v(x) = \lim_{t\to\infty}(d(x,\gamma_v(t)) - t)$, the corresponding *Busemann function* for γ_v. The level sets, ${b_v}^{-1}(t)$ are called *horospheres* of M.

A complete, simply connected Riemannian manifold without conjugate points is called *asymptotically harmonic* if the mean curvature of its horospheres is a universal constant, that is if its Busemann functions satisfy $\Delta b_v \equiv h$, $\forall v \in SM$, where h is a nonnegative constant. Then b_v is a smooth function on M for all v and all horospheres of M are smooth, simply connected hypersurfaces in M with constant mean curvature h.

For example, every simply connected, complete harmonic manifold without conjugate points is asymptotically harmonic.

For more details on this subject we refer to the discussion and to the references in [2]. Important result in this context are contained in [1], [3] and [6]. In [6] the following result was proved:

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Email : hemangi@math.iisc.ernet.in.

Theorem 1.1. *Let (M, g) be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. We show that M is a hyperbolic manifold of constant sectional curvature $\frac{-h^2}{4}$, provided M is asymptotically harmonic of constant $h > 0$.*

In this paper, we prove the above theorem for $h = 0$.

Theorem 1.2. *Let (M, g) be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. If M is asymptotically harmonic of constant $h = 0$, then M is flat.*

2 Proof of Theorem 1.2 :

We show that our proof in [6] can be used in case when $h = 0$. In fact, the final argument of the proof of the Theorem 1.2, viz. Lemma 2.5, is on similar lines as the corresponding lemma in [6]. Therefore, we recall some notations which were already used in the paper. Our general assumption is that M is 3-dimensional, has no conjugate points and is asymptotically harmonic with constant $h = 0$. For $v \in SM$ and $x \in v^\perp$, let

$$u^+(v)(x) = \nabla_x \nabla b_{-v} \quad \text{and} \quad u^-(v)(x) = -\nabla_x \nabla b_v.$$

Thus $u^\pm(v) \in \text{End }(v^\perp)$. The endomorphism fields $u^\pm$ satisfy the Riccati equation along the orbits of the geodesic flow $\varphi^t : SM \to SM$.
Thus if $u^\pm(t) := u^\pm(\varphi^t v)$ and $R(t) := R(\cdot, \gamma_v'(t))\gamma_v'(t) \in \text{End}(\gamma_v'(t)^\perp)$, then

$$(u^\pm)' + (u^\pm)^2 + R = 0.$$

Lemma 2.1. *Let γ_v be a geodesic line, then $b_v^+ + b_v^- = 0$, where ${b_v}^\pm$ are two Busemann functions associated to γ_v.*

Proof. Let γ_v be a geodesic line. Then,

$$\begin{aligned} {b_v}^+(x) &= \lim_{t\to\infty} d(x, \gamma_v(t)) - t, \\ b_v^-(x) &= \lim_{t\to-\infty} d(x, \gamma_v(t)) + t \end{aligned}$$

are two *Busemann functions* associated to γ_v. As (M, g) is an asymptotically harmonic manifold with minimal horospheres, $\Delta b_v^\pm = h = 0$. Also,

$${b_v}^+(x) + b_v^-(x) = \lim_{t\to\infty} d(x, \gamma_v(t)) + d(x, \gamma_v(-t)) - 2t.$$

Hence, ${b_v}^+(x) + b_v^-(x) \geq 0$ for all x, by triangle inequality, and $({b_v}^+ + b_v^-)(\gamma_v(t)) = 0$ since γ_v is a line.
Thus, the minimum principle shows that $b_v^+ + b_v^- = 0$. □

Corollary 2.2. *$u^+(v) = u^-(v)$ for all $v \in SM$.*

Proof. From the definition of $b_v^{\pm}$, the equation $b_v^+ + b_v^- = 0$ is equivalent to $b_v^+(x) = -b_{-v}^+(x)$. Hence, $\nabla_x \nabla b_v = -\nabla_x \nabla b_{-v}$. Thus, from the definition of $u^{\pm}$ we get, $u^+ = u^-$. $\square$

Lemma 2.3. *For every point $p \in M$, there exists $v \in S_pM$ such that $u^+(v) = 0$. In particular, Ricci$(v, v) = 0$.*

Proof. It was proved in [4] that: If (M, g) is an asymptotically harmonic manifold, then the map $B : SM \to C^\infty(M)$, given by $B(v) = b_v$ is continuous with respect to the C^∞ topology on the range space. In particular, the map $v \to \nabla^2 b_v$ is continuous on S_pM. Therefore, eigen values of $u^+(v)$ vary continuously with $v \in S_pM$.
Fix $p \in M$. Note that as dimension of M is 3, $u^+(v)$ is 2×2 traceless symmetric matrix, $u^+(v) \sim diag\ [\lambda^+(v), -\lambda^+(v)]$, where $\lambda^+(v)$ is an eigen value of $u^+(v)$. We may identify the tangent sphere S_pM with the standard 2-sphere S^2. Now consider the continuous map $f : S^2 \to \mathbb{R}^2$, defined by $f(v) = (\lambda^+(v), -\lambda^+(v))$. Then by Borsuk-Ulam theorem there exists $v \in S^2$ such that $f(v) = f(-v)$. Therefore,

$$\lambda^+(v) = \lambda^+(-v). \tag{1}$$

But, by definition of $u^+(v)$ we have $u^+(-v) = -u^-(v)$. Hence equation (1) implies that

$$\lambda^+(v) = \lambda^+(-v) = -\lambda^-(v). \tag{2}$$

By Corollary, 2.2 $u^+(v) = u^-(v)$. Thus,

$$\lambda^+(v) = \lambda^-(v). \tag{3}$$

Therefore, equations (2) and (3) imply that $\lambda^+(v) = 0$, and in turn $u^+(v) = 0$. Now Ricci$(v, v) = 0$ follows from the Riccati equation.

$\square$

Remark : It was proved in [5] that: If (M, g) is a non-compact, complete and simply connected manifold without conjugate points. If M is two or three dimensional, then, at each point $p \in M$ there exists a unit tangent vector u such that there is no focal point of p along the ray $\gamma_u(t), 0 < t < \infty$. In particular, the above lemma follows from this theorem.

Lemma 2.4. *For all $v \in SM$ we have* $\mathrm{Ric}(v, v) \leq 0$.

Proof. The Riccati equation for $t \mapsto u^+(t)$ implies $(u^+)' + (u^+)^2 + R = 0$. Hence, $\mathrm{tr}(u^+)^2 + \mathrm{tr}\, R = 0$. Thus, $\mathrm{Ric}(v, v) = -({\lambda_1}^2(v) + {\lambda_2}^2(v))$. By hypothesis $\lambda_1(v) + \lambda_2(v) = 0$, hence ${\lambda_1}^2(v) + {\lambda_2}^2(v) = 2{\lambda_1}^2(v) = 2{\lambda_2}^2(v)$. Consequently, $\mathrm{Ric}(v, v) \leq 0$. $\square$

Lemma 2.5. *The sectional curvature K of M satisfies $K \leq 0$.*

Proof. Let $p \in M$, and let v be the vector in Lemma 2.3. Take $e_1 = v$, and let e_2 and e_3 be unit vectors orthogonal to e_1 so that $\{e_1, e_2, e_3\}$ forms an orthonormal basis of T_pM. Then $\{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3\}$ forms an orthonormal basis of $\Lambda^2 T_pM$. We want to show that the curvature operator, considered as map $R : \Lambda^2 T_pM \to \Lambda^2 T_pM$, $\langle R(X \wedge Y), V \wedge W\rangle = \langle R(X, Y)W, V\rangle$ is diagonal in this basis.
Since, $\mathrm{Ric}(e_1, e_1) = 0, K(e_1, e_2) + K(e_1, e_3) = 0$. Now two cases arise:
(i) $K(e_1, e_2) = K(e_1, e_3) = 0$.
Then $K(e_2, e_3) \leq 0$ as $\mathrm{Ric}(e_2, e_2) \leq 0$, where $K(v, w)$ denotes the sectional curvature of the plane spanned by v and w. We will prove below that

$$\langle R(e_1, e_3)e_3, e_2\rangle = 0 \quad \text{and} \quad \langle R(e_1, e_2)e_2, e_3\rangle = 0. \tag{4}$$

Assuming this for a moment, it follows that $R(e_1 \wedge e_3) \perp \mathrm{span}\{e_1 \wedge e_2,\ e_2 \wedge e_3\}$ and $R(e_1 \wedge e_2) \perp \mathrm{span}\{e_1 \wedge e_3, e_2 \wedge e_3\}$. Hence,

$$R(e_1 \wedge e_2) = 0\ e_1 \wedge e_2 = 0 \text{ and } R(e_1 \wedge e_3) = 0\ e_1 \wedge e_3 = 0.$$

Since $e_1 \wedge e_2$ and $e_1 \wedge e_3$ are eigenvectors of R, also $e_2 \wedge e_3$ is an eigenvector and we obtain

$$R(e_2 \wedge e_3) = K(e_2, e_3)\ e_2 \wedge e_3.$$

Thus the curvature operator is diagonal in the basis $\{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3\}$ and all eigenvalues are ≤ 0, which proves the result.

It remains to show (4). Consider for $t \in (-\varepsilon, \varepsilon)$ the vectors $v_t = \cos t e_1 + \sin t e_2$. Then,

$$f(t) := \mathrm{Ric}(v_t, v_t) = K(v_t, e_3) + K(v_t, -e_1 \sin t + e_2 \mathrm{cos} t)$$
$$= K(e_1, e_2) + \sin^2 t\ K(e_2, e_3) + \cos^2 t\ K(e_1, e_3) + \sin 2t\ \langle R(e_1, e_3)e_3, e_2\rangle .$$

By Lemma 2.4 $f(0) = \mathrm{Ric}(v, v) = 0$ is maximal and hence $f'(0) = 0$. This implies the first equation in (4). If we replace e_2 by e_3 in the above computation we obtain the second equation.

In the sequel we use the following notations.

Let $v \in SM$, then $K(v) := \{K(\tau) | v \in \tau\} \subset \mathbb{R}$ is the set of sectional curvatures $K(\tau)$, where τ is a plane in the tangent space containing the given vector v. We write $a \leq K(v) \leq b$, if a is a lower and b an upper bound for the set $K(v)$.

Now consider case (ii)
(ii) $K(e_1, e_3) = -K(e_1, e_2)$. Let $K(e_1, e_3) = a > 0$. Then $K(e_1, e_2) = -a < 0$ and $K(e_2, e_3) < 0$, as $\mathrm{Ric}(e_3, e_3) \leq 0$. Hence, $K(e_2) < 0$. By continuity of $K(\tau)$, it follows that $K(\gamma'(t)) < 0$, where $\gamma : \mathbb{R} \to M$ is the geodesic

with $\gamma'(0) = e_2$. It is known that in a strictly negatively curved manifold horospheres are convex i.e. $u^+(v) < 0,\ u^-(v) > 0$. By the same argument it follows that $u^+(\gamma'(t)) = 0$, as $u^\pm$ is a traceless matrix. Hence, by case (i) it follows that $K \leq 0$. Thus this case can't occur and we are through. $\square$

Finally we come to the

Proof. of Theorem 1.2
Lemma 2.5 implies that $K_M \leq 0$. By standard comparison geometry we obtain $\lambda_1(v) \geq 0$ and $\lambda_2(v) \geq 0$. Now $\lambda_1 + \lambda_2 = 0$ implies that $\lambda_1 = \lambda_2 = 0$. Hence, $u^+(v) \equiv 0$ and therefore $R(x, v)v = 0, \forall v$ and $\forall x \in v^\perp$. Thus, $K_M \equiv 0$. $\square$

References

[1] G. Besson, G. Courtois, S. Gallot Entropies et rigidités des espaces localement symétriques de courbure strictement négative, Geom. Funct. Anal. **5** (1995), 731-799.

[2] J. Heber, G. Knieper, H. Shah Asymptotically harmonic spaces in dimension 3, Proc. Amer. Math. Soc. **135** (2007), 845-849.

[3] G. Knieper Spherical means on compact Riemannian manifolds of negative curvature, Differential Geom. Appl. **4** (1994), 361-390.

[4] A. Ranjan and H. Shah Busemann functions in a harmonic manifold, Geom. Dedicata **101** (2003), 167-183.

[5] A. Ranjan and H. Shah, Convexity of spheres in a manifold without conjugate points, Proc. Indian Acad. Sci. Math. Sci. **112** (2002), 595-599.

[6] V. Schroeder and H. Shah On 3-dimensional asymptotically harmonic manifolds, Arch. Math. **90** (2008), 275-278.